

DENSITY OF EIGENVALUES AND ITS PERTURBATION INVARIANCE IN UNITARY ENSEMBLES OF RANDOM MATRICES

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ABSTRACT. We generally study the density of eigenvalues in unitary ensembles of random matrices from the recurrence coefficients with regularly varying conditions for the orthogonal polynomials. First we calculate directly the moments of the density. Then, by studying some deformation of the moments, we get a family of differential equations of first order which the densities satisfy (see Theorem 1.2), and give the densities by solving them. Further, we prove that the density is invariant after the polynomial perturbation of the weight function (see Theorem 1.5).

1. INTRODUCTION AND MAIN RESULTS

Statistical behavior of eigenvalues of random matrices was first studied by E.P. Wigner in 1950s to get an information about spectra of heavy nuclei, and the famous semicircle-law was found then. In 1960s random matrices were intensively developed by E.P. Wigner, F.J. Dyson, M.L. Mehta and others for a better understanding of statistical behavior of energy levels in nuclear Physics, see [Po] as a collection of early papers. Later more and more importance was gained in other areas of Physics and Mathematics (see [Me] for a comprehensive introduction, or J. Phys. A 36, 2003).

It is of special importance to observe that the eigenvalue distribution of unitary invariant ensembles which can be described in [FK] by weight functions from the families of classical orthogonal polynomials. Such ensembles can be defined by the probability distribution on a space H_N of N -order Hermitian matrices as

$$P(dA) = F(A) \prod_{j=1}^N da_{jj} \prod_{j < k} d\Re a_{jk} d\Im a_{jk} \quad (1.1)$$

where F is an integrable nonnegative class function on H_N .

Using Proposition 2 in [HZ] we obtain

$$\int_{H_N} F(A) dA = Z_N \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} F(\text{diag}(x_1, \dots, x_N)) \prod_{1 \leq j < k \leq N} (x_j - x_k)^2 dx_1 \dots dx_N \quad (1.2)$$

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where

$$Z_N = \frac{\pi^{k(k-1)/2}}{k!(k-1)!\cdots 1!}. \quad (1.3)$$

Especially when

$$F(\text{diag}(x_1, \dots, x_N)) = Z_N^{-1} \prod_{j=1}^N \omega(x_j) \chi_I(x_j), \quad (1.4)$$

one introduces the joint distribution function for the eigenvalues

$$P_N(x_1, x_2, \dots, x_N) = \prod_{j=1}^N \omega(x_j) \prod_{1 \leq j < k \leq N} (x_j - x_k)^2 \quad (1.5)$$

where $I = (\alpha, \beta)$, $-\infty \leq \alpha < \beta \leq +\infty$ and $\omega(x)$ is a weight function on the interval I all finite moments of which exist.

Remark 1.1 (1.5) was imposed on a definition of a matrix ensemble in [Me], and obtained in [TW] for

$$F(A) = \exp(-\text{Tr}(V(A))) \quad (1.6)$$

where $V(x)$ is a real-valued function such that $\omega(x) = \exp(-V(x))$ defines a weight function. In addition, it is easy to draw the conclusion similar to (1.5) for the orthogonal and symplectic ensembles.

Remark 1.2 The appearance of the characteristic function χ_I of the set I means that we just consider sub-ensemble of Hermitian matrices space whose eigenvalues are all in I . Moreover, I can be given by the union of some disjoint intervals and then $\omega(x)$ is the associated weight function.

From (1.5) the n -point correlation function is defined in [Me] by

$$R_n(x_1, \dots, x_n) = \frac{N!}{(N-n)!} \int_I \cdots \int_I P_N(x_1, x_2, \dots, x_N) dx_{n+1} \cdots dx_N. \quad (1.7)$$

If we introduce the orthogonal polynomials

$$\int_I p_j(x) p_k(x) \omega(x) dx = \delta_{jk}, \quad j, k = 0, 1, \dots \quad (1.8)$$

and the associated functions

$$\varphi_k(x) = p_k(x) \sqrt{\omega(x)}, \quad (1.9)$$

then by the property of Vandermonde determinant the joint distribution function (1.5) reads:

$$\begin{aligned} P_N(x_1, x_2, \dots, x_N) &= \frac{1}{N!} (\det[\varphi_{j-1}(x_k)]_{j,k=1}^N)^2 \\ &= \frac{1}{N!} \det[K_N(x_j, x_k)]_{j,k=1}^N \end{aligned} \quad (1.10)$$

where

$$K_N(x, y) = \sum_{j=0}^{N-1} \varphi_j(x) \varphi_j(y). \quad (1.11)$$

Because of the orthonormality of the $\varphi_j(x)$'s one can show just as in [Me, Chap.6] that

$$R_n(x_1, \dots, x_n) = \det[K_N(x_j, x_k)]_{j,k=1}^n. \quad (1.12)$$

It is of considerable interest to obtain the behavior of n -point correlation of (1.12) after some appropriate scaling in the limit of large N . In particular, putting $n = 1$, we get the density of eigenvalues (also called level density)

$$R_1(x) = K_N(x, x) = \sum_{j=0}^{N-1} \varphi_j^2(x) \quad (1.13)$$

and denote the normalized density by

$$\frac{1}{N} R_1(x) = \frac{1}{N} \sum_{j=0}^{N-1} \varphi_j^2(x). \quad (1.14)$$

Remark 1.3 For the eigenvalues x_1, x_2, \dots, x_N of a random Hermitian matrix A , the integral of the normalized counting function over an interval $\Delta = (c, d)$ is given by

$$\nu_N(\Delta) = \frac{1}{N} \int_{\Delta} \sum_{j=1}^N \delta(x - x_j) dx. \quad (1.15)$$

A calculation in [PS] shows that

$$\mathbb{E}(\nu_N)(\Delta) = \int_{\Delta} \frac{1}{N} R_1(x) dx \quad (1.16)$$

where $\mathbb{E}(\nu_N)(\Delta)$ denotes the expectation of $\nu_N(\Delta)$ with respect to the probability measure of (1.5).

Our motivation is to obtain the density in the scaling limit of large N from (1.14) and further study its polynomial-perturbation invariance. Before our results are stated we review some known results about the density.

As we know, Wigner in [Wig1, 2] not only got his famous semicircle law (here corresponding to the weight $\omega(x) = e^{-x^2}$ called Gauss unitary ensemble, denoting GUE)

$$\sigma(x) = \frac{2}{\pi} \sqrt{1 - x^2} \chi_{[-1,1]}(x), \quad (1.17)$$

but also invented the calculation method which had some independent interest as he thought. Afterwards, Bronk got the associated density for $\omega(x) = x^\rho e^{-x}$, $\rho > -1$, $0 < x < +\infty$ for Laguerre ensembles in [Br] and Lef for $\omega(x) = (1 - x)^a (1 + x)^b$, $a, b > -1$, $-1 < x < 1$ for Jacobian ensembles in [Lef]. Recently, an irradiative idea in [HT] was introduced by Haagerup and Thorbjørnsen to give a short proof of Wigner's semicircle law, using the Laplace transform of (1.14) and the property of Hermitian polynomials. In [Lef], Ledoux pushed forward the investigation in [HT] and in a slightly different way obtained Wigner's semicircle law, also the densities for Laguerre and Jacobian unitary ensembles but whose expressions appeared complicated. In addition, based on the spirit of statistical mechanics the method of equilibrium measure is used to obtain the density, mainly for the weight function $\omega(x) = \exp(-V(x))$,

$$V(x) = \gamma_{2m} x^{2m} + \dots + \gamma_0, \quad \gamma_{2m} > 0. \quad (1.18)$$

See [Jo], [DMK] or [De] for the equilibrium measure method. In a recent survey on orthogonal polynomial ensembles ([Kö]), König obtained semicircle law for GUE respectively by moment method and equilibrium measure method.

In this paper, we will generally deal with the weight function $\omega(x)$ on the interval I and obtain the density of eigenvalues $\sigma(x)$. In fact, Professor W. Van Assche told us that P. Nevai and W. Van Assche had shown that the density and zero distribution of the orthogonal polynomials were the same ([Va1], Theorem 5.3), see Remark 1.8 below. However, our method is to calculate directly the moments without other knowledge than three-term recurrence formula, and further the density was obtained using a different method. To our knowledge, the family of densities satisfying differential equations of first order is first obtained in the present paper.

It is a well known result that three-term recurrence formula holds for the orthogonal polynomials defined by (1.8)

$$xp_n(x) = a_{n+1}p_{n+1}(x) + b_np_n(x) + a_np_{n-1}(x), \quad n = 0, 1, \dots \quad (1.19)$$

where $a_n > 0, p_{-1}(x) = 0$. We will assume that there is a positive and non-decreasing sequence c_n such that

$$\lim_{n \rightarrow \infty} \frac{a_n}{c_n} = a > 0, \quad \lim_{n \rightarrow \infty} \frac{b_n}{c_n} = b. \quad (1.20)$$

An extra condition on the contraction sequence is that c_n is a regularly varying sequence with index $\lambda \geq 0$, i.e.

$$c_n = n^\lambda L(n) \quad (1.21)$$

where $L : (0, +\infty) \rightarrow (0, +\infty)$ is slowly varying, that is,

$$\lim_{x \rightarrow \infty} \frac{L(xt)}{L(x)} = 1, \quad \forall t > 0. \quad (1.22)$$

The condition of (1.21) was first introduced by W. Van Assche to study the asymptotics for orthogonal polynomials, see [Va2] as a general survey for the condition of (1.20) and [Fe, Chap.VIII] for regular functions (1.22).

Remark 1.4 Assuming $c(x)$ is a positive, non-decreasing differentiable function on the interval $(0, +\infty)$ with

$$\lim_{x \rightarrow +\infty} \frac{xc'(x)}{c(x)} = \lambda \geq 0, \quad (1.23)$$

then one easily knows that $c(x)$ can be represented as

$$c(x) = x^\lambda L(x). \quad (1.24)$$

A discrete version of this result is that (1.21) holds if

$$\lim_{n \rightarrow \infty} n \left(\frac{c_{n+1}}{c_n} - 1 \right) = \lambda \geq 0. \quad (1.25)$$

Remark 1.5 The conditions of (1.20) and (1.21) are connected close with the asymptotic problems of the orthogonal polynomials. The class of Freud weights plays a most close role on (1.21), for examples, $\omega(x) = \exp(-Q(x))$ where $Q(x)$ grows like a power at infinity, in particular,

$$Q(x) = \gamma_{2m}x^{2m} + \dots + \gamma_0, \quad \gamma_{2m} > 0 \quad (1.26)$$

with $\lambda = 1/(2m)$, see [DKMVZ] ;

$$\omega(x) = |x|^\beta e^{-|x|^\alpha}, \quad \beta > -1, \quad \alpha > 0 \quad (1.27)$$

with $\lambda = 1/\alpha$, see [LMS] or [Va1]. In addition, $\lambda = 1$ for Laguerre weights

$$\omega(x) = x^\alpha e^{-x}, \quad \alpha > -1 \quad (1.28)$$

and $\lambda = 0$ for Jacobi weights

$$\omega(x) = (1-x)^\alpha (1+x)^\beta, \quad \alpha, \beta > -1. \quad (1.29)$$

A classic result of E.A. Rakhmanov in [Si] asserts that if the weight ω on $[-1, 1]$ satisfying $\omega > 0$ a.e. on $[-1, 1]$, then $\omega(x)$ belongs to the Nevai-Blumenthal class, that is

$$\lim_{n \rightarrow \infty} a_n = 1/2, \quad \lim_{n \rightarrow \infty} b_n = 0. \quad (1.30)$$

Obviously, $\lambda = 0$ for the Nevai-Blumenthal class (see [Va2]). We strongly refer the reader to [Lub] for a recent survey for a wide variety of weights on finite or infinite intervals.

Now we can state our main results. In Section 2 we will introduce ascending, equilibrating and descending operators which describe the transforming of polynomials, and explicitly calculate the moments of the density. Then in Section 3 we consider a simple deformation of the moments for any given density and obtain a corresponding density determined by a differential equation with respect to the new moments. In section 4 using the results in Section 3 we give the proofs of Theorems 1.2 and 1.5 below.

First let us rescale the density of (1.14) by

$$\sigma_N(x) = \frac{c_N}{N} R_1(c_N x). \quad (1.31)$$

Note that $\sigma_N(x)$ is our main object and we will study its limit behavior.

Theorem 1.1 Denote the k th moment of the scaling density $\sigma_N(x)$ by $M_k^{(N)}$. Under the contraction conditions of (1.20) and (1.21), we have

$$\lim_{N \rightarrow \infty} M_k^{(N)} = M_k, \quad k = 0, 1, \dots \quad (1.32)$$

where

$$M_k = \frac{1}{1 + \lambda k} \left(\sum_{j=0}^{[k/2]} C_k^j C_{k-j}^j a^{2j} b^{k-2j} \right). \quad (1.33)$$

Remark 1.6 One sets $a = 1/2$ in the following since c_n can be chosen freely from some constant. Observe

$$\sum_{j=0}^{[k/2]} C_k^j C_{k-j}^j a^{2j} b^{k-2j} = L_0 \left(az + \frac{a}{z} + b \right)^k \quad (1.34)$$

where the operator $L_0(f)$ represents the constant term of Laurent series, using Cauchy contour integral, that is

$$L_0(f) = \frac{1}{2\pi i} \oint \frac{f(z)}{z} dz. \quad (1.35)$$

Theorem 1.2 A probability density $\sigma(x)$ with its k th moment M_k exists, and is uniquely determined by the following differential equation of first order

$$\sigma(x) - \lambda[x\sigma(x)]^{(1)} = \frac{1}{\pi} \frac{1}{\sqrt{1-(x-b)^2}} \chi_{I_b}, \quad (1.36)$$

with the following conditions

$$\sigma(x) \geq 0, \int_{-\infty}^{+\infty} \sigma(x) dx = 1 \quad (1.37)$$

where $I_b = (-1+b, 1+b)$ and χ_{I_b} is a characteristic function of I_b . Exactly, the support of $\sigma(x)$ can be restricted to a finite interval, that is, for $\lambda = 0$

$$\sigma(x) = \frac{1}{\pi} \frac{1}{\sqrt{1-(x-b)^2}} \quad (1.38)$$

while for $\lambda > 0$

$$\text{supp}(\sigma) = [B_1, B_2] \quad (1.39)$$

where

$$B_1 = \min\{b-1, 0\}, B_2 = \max\{b+1, 0\}. \quad (1.40)$$

One directly solves the equation of (1.36) and easily obtains

Corollary 1.3 For $b = 0$ and $m \in \mathbb{N}$, we have

(1) for $\lambda = 1/2m$

$$\sigma(x) = \frac{4}{\pi} \sqrt{1-x^2} \sum_{j=1}^m ((2x)^{2m-2j} C_{2j-2}^{j-1}) / C_{2m}^m \quad (1.41)$$

and

(2) for $\lambda = 1/(2m-1)$

$$\sigma(x) = \frac{mC_{2m}^m}{2\pi} \left(\left(\frac{x}{2}\right)^{2m-2} \ln \frac{1+\sqrt{1-x^2}}{|x|} + \frac{\sqrt{1-x^2}}{2} \sum_{j=1}^{m-1} \frac{1}{jC_{2j}^j} \left(\frac{x}{2}\right)^{2m-2-2j} \right). \quad (1.42)$$

For $b = -1$ and $q \in \mathbb{N} \cup \{0\}$, we have

(3) for $\lambda = 1/(q+1)$

$$\sigma(x) = \frac{q+1}{\pi} \left(\frac{x}{2}\right)^q \sum_{j=0}^q \frac{C_q^j}{1+2j} \left(\sqrt{\frac{2-x}{x}}\right)^{1+2j} \quad (1.43)$$

and

(4) for $\lambda = 1/(q + \frac{1}{2})$

$$\sigma(x) = \frac{2q+1}{4\pi} \left(\left(\frac{x}{8}\right)^{q-\frac{1}{2}} \ln \left(\sqrt{\frac{2-x}{x}} + \sqrt{\frac{2}{x}} \right) + \sqrt{\frac{2-x}{x}} \sum_{j=1}^q \frac{1}{jC_{2j}^j} \left(\frac{x}{8}\right)^{q-j} \right) C_{2q}^q. \quad (1.44)$$

Remark 1.7 For the weight $\omega(x) = \exp(-V(x))$ from (1.18) which is corresponding with $\lambda = 1/2m$, the density was given in [Jo] in the form of $r(x)\sqrt{(x-x_1)(x_2-x)}$. Here $x_1 < x_2$ and $r(x)$ is a polynomial of degree $2m-2$ depending on $V(x)$; when $\lambda = 1$ and $b = 0$, it was obtained in [CI] using Mathematica for Meixner-Pollaczek polynomials. It is just the case where $\lambda = 1$ and $b = -1$ for Laguerre weights of (1.28), see [Sze]. The authors believe that the densities in Corollary 1.3, especially for small integrals m and q , may appear in some other random matrix models.

From Theorems 1.1 and 1.2, it is obvious that

Theorem 1.4 Denote $\sigma_N(x)$ and $\sigma(x)$ as above, then

$$\sigma_N(x) \xrightarrow{W} \sigma(x) \quad (1.45)$$

where W means in the weak sense.

Remark 1.8 Let $x_{1,n}, x_{2,n}, \dots, x_{n,n}$ be zeros of p_n . P. G. Nevai and J. S. Dehesa in [ND] also got the same moments of zero distribution under the contraction condition of (1.20) (this fact was pointed out by W. Van Assche in [Va2]), namely,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n \left(\frac{x_{j,n}}{c_n} \right)^k = M_k. \quad (1.46)$$

A beautiful probabilistic interpretation in [Va1] shows M_k is the k th moment of Nevai-Ullman measure. In addition, Nevai and Van Assche proved that weak convergence of zero distribution implied some weak convergence to the same probability measure for Christoffel functions, which are defined by $(\sum_{j=0}^{N-1} p_j^2(x))^{-1}$ (see Theorem 5.3 in [Va1]). It is obvious that the density of eigenvalues and zero distribution are the same according to (1.46) and Theorem 1.1, thus we give a new proof.

Let $p(x)$ be a fixed l th order polynomial, and we consider a new weight function

$$\hat{\omega}(x) = p^2(x)\omega(x) \quad (1.47)$$

with

$$\int_I \hat{p}_j(x) \hat{p}_k(x) \hat{\omega}(x) dx = \delta_{jk}, \quad j, k = 0, 1, \dots \quad (1.48)$$

Associated 1-point correlation function is given by

$$\hat{R}_1(x) = \sum_{j=0}^{N-1} \hat{p}^2(x) p^2(x) \omega(x). \quad (1.49)$$

Under the same scaling we write

$$\hat{\sigma}_N(x) = \frac{c_N}{N} \hat{R}_1(c_N x). \quad (1.50)$$

Calculate the moments of (1.50) for the new weight function, we find

Theorem 1.5 Denote the k th moments of $\hat{\sigma}_N(x)$ and $\sigma_N(x)$ by $M_k^{(N)}$ and $\hat{M}_k^{(N)}$, respectively. Under the contraction conditions of (1.20) and (1.21), we have

$$\lim_{n \rightarrow \infty} \hat{M}_k^{(N)} = \lim_{n \rightarrow \infty} M_k^{(N)} = M_k, \quad k = 0, 1, \dots \quad (1.51)$$

Namely,

$$\hat{\sigma}_N(x) \xrightarrow{W} \sigma(x) \quad (1.52)$$

where $\sigma(x)$ is the density determined by (1.36) and (1.37).

2. CALCULATION OF THE MOMENTS

By the recursion formula of (1.19), we regard the multiplication by x as an operator A_x , and it can be represented as

$$A_x = A_+ + A_0 + A_- \quad (2.1)$$

where A_+ , A_0 and A_- are called ascending, equilibrating and descending operators respectively, defined by

$$A_+ p_n(x) = a_{n+1} p_{n+1}(x), \quad A_0 p_n(x) = b_n p_n(x), \quad A_- p_n(x) = a_n p_{n-1}(x). \quad (2.2)$$

Thus we calculate the k th moment of $\sigma_N(x)$ using (1.31), (2.1) and (2.2) as follows:

$$\begin{aligned} M_k^{(N)} &= \int x^k \sigma_N(x) dx \\ &= \frac{1}{N (c_N)^k} \sum_{j=0}^{N-1} \int_I x^k p_j^2 \omega(x) dx \\ &= \frac{1}{N (c_N)^k} \sum_{j=0}^{N-1} \langle x^k p_j, p_j \rangle_{L^2(\omega)} \\ &= \frac{1}{N (c_N)^k} \sum_{j=0}^{N-1} \langle (A_+ + A_0 + A_-)^k p_j, p_j \rangle_{L^2(\omega)}. \end{aligned} \quad (2.3)$$

Let Λ_k^q be a set composed of those terms in the expansion of $(A_+ + A_0 + A_-)^k$, in which the operators A_+ and A_- exactly appear q times respectively. Note that $\langle T p_j, p_j \rangle_{L^2(\omega)} = 0$ for $T \notin \bigcup_q \Lambda_k^q$, then one obtains the following lemma:

Lemma 2.1

$$M_k^{(N)} = \frac{1}{N (c_N)^k} \sum_{j=0}^{N-1} \sum_{q=0}^{[k/2]} \sum_{T \in \Lambda_k^q} \langle T p_j, p_j \rangle_{L^2(\omega)}. \quad (2.4)$$

For convenience of the following calculation, we introduce one property of the regular varying functions.

Lemma 2.2 Let $c(x)$ be a positive and non-decreasing function on $(0, +\infty)$, and

$$c(x) = x^\lambda L(x) \quad (2.5)$$

where $L(x)$ satisfies

$$\lim_{x \rightarrow +\infty} \frac{L(xt)}{L(x)} = 1, \quad \forall t > 0. \quad (2.6)$$

Then

$$\lim_{n \rightarrow \infty} \frac{\frac{1}{n} \int_1^n (c(x))^k dx}{(c(n))^k} = \frac{1}{1 + \lambda k}. \quad (2.7)$$

Proof By Lebesgue's dominated theorem and exchanging limits and integrals, it is easy to prove using (2.5) and (2.6). \square

Now we give a proof of Theorem 1.1 by two cases of b .

Case 1 $b \neq 0$.

Assume that $b > 0$ for convenience. Write

$$\frac{a_n}{c_n} = a(1 + \xi_n), \quad \frac{b_n}{c_n} = b(1 + \eta_n). \quad (2.8)$$

If one writes for $j > k$

$$u_j = \max_{j-k \leq m \leq j+k} \{|\xi_m|, |\eta_m|\}, \quad (2.9)$$

then

$$\lim_{j \rightarrow \infty} u_j = 0. \quad (2.10)$$

Thus we can suppose $u_j < 1, \forall j = 0, 1, \dots$ and by the definition of u_j one obtains

$$\max_{j-k \leq m \leq j+k} \{a_m\} \leq a c_{j+k}(1 + u_j), \quad \max_{j-k \leq m \leq j+k} \{b_m\} \leq b c_{j+k}(1 + u_j). \quad (2.11)$$

Furthermore, for $T \in \Lambda_k^q$ we have

$$\begin{aligned} \left\langle T p_j, p_j \right\rangle_{L^2(\omega)} &\leq \left(\max_{j-k \leq m \leq j+k} \{a_m\} \right)^{2q} \left(\max_{j-k \leq m \leq j+k} \{b_m\} \right)^{k-2q} \\ &\leq a^{2q} b^{k-2q} (c_{j+k})^k (1 + u_j)^k. \end{aligned} \quad (2.12)$$

Summing by q and j , we get

$$M_k^{(N)} \leq \left(\sum_{q=0}^{[k/2]} C_k^q C_{k-q}^q a^{2q} b^{k-2q} \right) \frac{\frac{1}{N} \sum_{j=0}^{N-1} (c_{j+k})^k (1 + u_j)^k}{(c_N)^k}. \quad (2.13)$$

By Cauchy-Maclaurin summation formula and Lemma 2.2 one obtains

$$\begin{aligned} \lim_{N \rightarrow \infty} \frac{\frac{1}{N} \sum_{j=0}^{N-1} (c_{j+k})^k (1 + u_j)^k}{(c_N)^k} &= \lim_{N \rightarrow \infty} \frac{\frac{1}{N} \sum_{j=0}^{N-1} (c_{j+k})^k}{(c_N)^k} \\ &= \lim_{N \rightarrow \infty} \frac{\frac{1}{N} \sum_{j=1}^N (c_j)^k}{(c_N)^k} \\ &= \lim_{N \rightarrow \infty} \frac{\frac{1}{N} \int_1^N (c(x))^k dx}{(c(N))^k} \\ &= \frac{1}{1 + \lambda k}. \end{aligned} \quad (2.14)$$

Analogously, for $j > k$, one obtains

$$M_k^{(N)} \geq \left(\sum_{q=0}^{[k/2]} C_k^q C_{k-q}^q a^{2q} b^{k-2q} \right) \frac{\frac{1}{N} \sum_{j=k}^{N-1} (c_{j-k})^k (1 - u_j)^k}{(c_N)^k} \quad (2.15)$$

and

$$\lim_{N \rightarrow \infty} \frac{\frac{1}{N} \sum_{j=k}^{N-1} (c_{j-k})^k (1 - u_j)^k}{(c_N)^k} = \frac{1}{1 + \lambda k}. \quad (2.16)$$

Combining (2.13) — (2.16) we complete the proof of this case. \square

Case 2 $b = 0$.

Similarly, write

$$\frac{a_n}{c_n} = a(1 + \xi_n), \quad \frac{b_n}{c_n} = \eta_n \quad (2.17)$$

and

$$v_j = \max_{j-k \leq m \leq j+k} \{|\xi_m|, |\eta_m|\}, \quad (2.18)$$

then

$$\lim_{j \rightarrow \infty} v_j = 0. \quad (2.19)$$

Thus we can suppose $v_j < 1, \forall j = 0, 1, \dots$. For $0 \leq j - k \leq m \leq j + k$, one obtains

$$a c_{j-k} (1 - v_j) \leq a_m \leq a c_{j+k} (1 + v_j), \quad |b_m| \leq c_{j+k} v_j. \quad (2.20)$$

Now we give an estimation of $M_k^{(N)}$ when k is odd. For $T \in \Lambda_k^q$, by (2.20) we have

$$\begin{aligned} \left| \left\langle T p_j, p_j \right\rangle_{L^2(\omega)} \right| &\leq a^{2q} v_j^{k-2q} (c_{j+k})^k (1 + v_j)^{k-2q} \\ &\leq a^{2q} v_j^{k-2q} (c_{j+k})^k (1 + v_j)^k. \end{aligned} \quad (2.21)$$

If denoting

$$\bar{v}_j = \sum_{q=0}^{[k/2]} C_k^q C_{k-q}^q a^{2q} v_j^{k-2q}, \quad (2.22)$$

then

$$\lim_{j \rightarrow \infty} \bar{v}_j = 0. \quad (2.23)$$

Thus by the Stoltz formula one obtains

$$\left| M_k^{(N)} \right| \leq \frac{\frac{1}{N} \sum_{j=0}^{N-1} (c_{j+k})^k (1 + v_j)^k \bar{v}_j}{(c_N)^k} \longrightarrow 0. \quad (2.24)$$

When k is even. For $T \in \Lambda_k^q, q \neq k/2$,

$$\left| \left\langle T p_j, p_j \right\rangle_{L^2(\omega)} \right| \leq a^{2q} v_j^{k-2q} (c_{j+k})^k (1 + v_j)^k \quad (2.25)$$

and for $T \in \Lambda_k^{k/2}, j > k$,

$$a^k (c_{j-k})^k (1 - v_j)^k \leq \left\langle T p_j, p_j \right\rangle_{L^2(\omega)} \leq a^k (c_{j+k})^k (1 + v_j)^k. \quad (2.26)$$

If one writes

$$\overline{M}_k^{(N)} = \frac{1}{N (c_N)^k} \sum_{j=0}^{N-1} \sum_{q < k/2} \sum_{T \in \Lambda_k^q} \left\langle T p_j, p_j \right\rangle_{L^2(\omega)} \quad (2.27)$$

and

$$\widetilde{M}_k^{(N)} = \frac{1}{N (c_N)^k} \sum_{j=0}^{N-1} \sum_{T \in \Lambda_k^{k/2}} \left\langle T p_j, p_j \right\rangle_{L^2(\omega)}, \quad (2.28)$$

then

$$M_k^{(N)} = \overline{M}_k^{(N)} + \widetilde{M}_k^{(N)}. \quad (2.29)$$

Similar to the case where k is odd, by (2.25) one obtains

$$\lim_{N \rightarrow \infty} \overline{M}_k^{(N)} = 0 \quad (2.30)$$

and similar to the case where $b \neq 0$, by (2.26) one obtains

$$\lim_{N \rightarrow \infty} \widetilde{M}_k^{(N)} = \frac{1}{1 + \lambda k} C_k^{k/2} a^k. \quad (2.31)$$

Combining (2.24), (2.29), (2.30) and (2.31) we complete the proof of this case where $b = 0$. \square

Therefore, Theorem 1.1 has been proved.

3. DEFORMATION OF THE MOMENTS

Following Wigner's original papers [Wig1] and [Wig2], to obtain the distribution of eigenvalues in random matrix models, a standard procedure is first to calculate the moments of all orders after some appropriate scaling of eigenvalues and then to determine an explicit distribution function of the moments. In this section we discuss some deformations of the moments of a given density function and determine the corresponding density function with respect to the new moments, and then give an application in the following section. A natural question: which kind of deformations of the moments has a corresponding distribution function? We will make an interesting try.

Let $f(x)$ be a continuous density function on the interval $I = (\alpha, \beta)$, $-\infty \leq \alpha < \beta \leq +\infty$, and its moments

$$m_k = \int_I x^k f(x) dx, \quad k = 0, 1, 2, \dots \quad (3.1)$$

exist. Now consider the following deformation of moments with a parameter $\lambda > 0$,

$$M_k = \frac{1}{1 + \lambda k} m_k, \quad k = 0, 1, 2, \dots \quad (3.2)$$

Question: When is there a unique density function $\sigma(x)$ which is a solution of the moment problem of (3.2), i.e.

$$\int_{-\infty}^{+\infty} x^k \sigma(x) dx = M_k, \quad k = 0, 1, 2, \dots, \quad (3.3)$$

and further how to determine $\sigma(x)$ from the given density $f(x)$?

Note that two density functions $f(x) = g(x)$ are said to be equal if their distribution functions $\int_{-\infty}^x f(s) ds$ and $\int_{-\infty}^x g(s) ds$ are equal at their points of continuity.

To make sure that the density is unique we assume the moments $m_k, k = 0, 1, 2, \dots$ satisfy Carleman's condition

$$\sum_{k=0}^{\infty} m_{2k}^{-1/2k} = \infty. \quad (3.4)$$

It is obvious that the moments M_k also satisfy Carleman's condition

$$\sum_{k=0}^{\infty} M_{2k}^{-1/2k} = \infty. \quad (3.5)$$

Due to Carleman's famous theorem (see [ST] or [Fe]) (3.5) assures that the moment problem of (3.2) is determined, i.e. numbers M_k determine a unique density whose k th moment is M_k .

To prove that the moment problem of (3.2) has a solution $\sigma(x)$ whose spectrum support $\text{supp}\sigma(x)$ is to be contained in the interval J , given in advance, we first introduce an important theorem. Let $P(u)$ be any polynomial in u ,

$$P(u) = \sum_k x_k u^k \quad (3.6)$$

where numbers x_k are real constants. Introduce the functional $\mu(P)$ defined by

$$\mu(P) = \sum_k M_k x_k. \quad (3.7)$$

Theorem 3.1 ([ST], Theorem 1.1) A necessary and sufficient condition that the moment problem defined by the sequence of moments M_k shall have a solution on J is that the functional $\mu(P)$ be non-negative, that is

$$\mu(P) \geq 0, \quad \text{whenever } P(u) \geq 0 \text{ on } J. \quad (3.8)$$

Note that Theorem 3.1 can be applied to derive explicit necessary and sufficient conditions by a special choice of J , which depend on representations of non-negative polynomials on J . In particular, it is a well-known fact (see [PóS]) any polynomial $P(u) \geq 0$ for all real u can be presented as

$$P(u) = P_1(u)^2 + P_2(u)^2 \quad (3.9)$$

where $P_1(u)$ and $P_2(u)$ are polynomials with real coefficients.

If we take for $P(u)$ the particular polynomial $P(u) = (x_0 + x_1 u + \cdots + x_n u^n)^2$, we have

$$\mu(P) = \sum_{j,k=0}^n M_{j+k} x_j x_k. \quad (3.10)$$

From the theory of quadratic forms and (3.9) it is well known that the conditions of (3.8) are equivalent to

$$|M_{j+k}|_{j,k=0}^n > 0, \quad n = 0, 1, 2, \dots \quad (3.11)$$

if the spectrum of the solution is not reducible to a finite set of points.

Now we can state our theorem as follows:

Theorem 3.2 A probability density $\sigma(x)$ on the real axis with its k th moment M_k of (3.2) exists, and is uniquely determined by the differential equation of first order

$$\sigma(x) - \lambda[x\sigma(x)]^{(1)} = f(x)\chi_I \quad (3.12)$$

with the following conditions

$$\sigma(x) \geq 0, \quad \int_{-\infty}^{+\infty} \sigma(x) dx = 1. \quad (3.13)$$

Proof. It is sufficient to prove (3.11) if $(M_{j+k})_{j,k=0}^n$ is a positive definite matrix. Taking $f(x)\chi_I$ for the solution of the moments m_k on the real axis, we have

$$|m_{j+k}|_{j,k=0}^n > 0, \quad n = 0, 1, 2, \dots, \quad (3.14)$$

that is,

$$\Delta_n = (m_{j+k})_{j,k=0}^n \text{ is a positive definite matrix, } n = 0, 1, 2, \dots. \quad (3.15)$$

The formula of (3.2) shows that

$$(M_{j+k})_{j,k=0}^n = \Delta_n * \Lambda_n \quad (3.16)$$

where $*$ represents Schur product and

$$\Lambda_n = \left(\frac{1}{1 + \lambda(j+k)} \right)_{j,k=0}^n, \quad n = 0, 1, 2, \dots. \quad (3.17)$$

Thus by the property of Schur product it is sufficient to prove that Λ_n is a positive definite matrix. Note that

$$\frac{1}{1 + \lambda k} = \int_0^1 t^{\lambda k} dt, \quad (3.18)$$

then one can obtain

$$\det \Lambda_n = \frac{1}{(n+1)!} \int_0^1 \cdots \int_0^1 \prod_{0 \leq j < k \leq n} (t_j^\lambda - t_k^\lambda)^2 \prod_{j=0}^n dt_j > 0. \quad (3.19)$$

To derive the equation of (3.12) we make a Fourier transform and write

$$H(t) = \int_{-\infty}^{+\infty} e^{itx} \sigma(x) dx, \quad F(t) = \int_I e^{itx} f(x) dx. \quad (3.20)$$

Thus,

$$\begin{aligned} \int_{-\infty}^{+\infty} itx e^{itx} \sigma(x) dx &= \sum_{k=0}^{\infty} \frac{(it)^{k+1}}{k!} M_{k+1} \\ &= \sum_{k=0}^{\infty} \frac{(it)^{k+1}}{(k+1)!} (k+1) M_{k+1} \\ &= \sum_{k=0}^{\infty} \frac{(it)^k}{k!} k M_k. \end{aligned} \quad (3.21)$$

Obviously,

$$\begin{aligned} \int_{-\infty}^{+\infty} (1 + \lambda itx) e^{itx} \sigma(x) dx &= \sum_{k=0}^{\infty} \frac{(it)^k}{k!} (1 + \lambda k) M_k \\ &= \sum_{k=0}^{\infty} \frac{(it)^k}{k!} m_k \\ &= \int_I e^{itx} f(x) dx. \end{aligned} \quad (3.22)$$

Namely,

$$H(t) + \lambda t H'(t) = F(t). \quad (3.23)$$

We also make an inverse Fourier transform. Combining

$$\begin{aligned} \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-itx} t H'(t) dt &= -\frac{1}{2\pi i} \left[\int_{-\infty}^{+\infty} e^{-itx} H'(t) dt \right]^{(1)} \\ &= -\frac{1}{2\pi i} \left[ix \int_{-\infty}^{+\infty} e^{-itx} H(t) dt \right]^{(1)} \\ &= -[x\sigma(x)]^{(1)} \end{aligned} \quad (3.24)$$

and (3.23) one obtains the differential equation on real axis

$$\sigma(x) - \lambda[x\sigma(x)]^{(1)} = f(x)\chi_I \quad (3.25)$$

whose possible singular points are 0, α and β (if α and β are finite numbers). \square

Note that if α or β is finite we have an exact information about the spectrum support J of $\sigma(x)$.

Case 1. $-\infty < \alpha \leq 0$, $\beta = +\infty$ (or $\alpha = -\infty$, $0 \leq \beta < +\infty$ similarly). $J = I$.

Any non-negative polynomial $P(u)$ on $I = (\alpha, +\infty)$ can be represented by

$$P(u) = P_1(u)^2 + P_2(u)^2 + (u - \alpha)(P_3(u)^2 + P_4(u)^2) \quad (3.26)$$

where $P_1(u)$, $P_2(u)$, $P_3(u)$ and $P_4(u)$ are polynomials with real coefficients.

An analogous procedure to the above arguments of (3.9) — (3.11) one obtains the conditions of (3.8) are equivalent to the matrices

$$(M_{j+k})_{j,k=0}^n, \quad n = 0, 1, 2, \dots \quad (3.27)$$

and

$$(M_{j+k+1} - \alpha M_{j+k})_{j,k=0}^n, \quad n = 0, 1, 2, \dots \quad (3.28)$$

are positive definite.

Note that

$$M_{k+1} - \alpha M_k = \frac{m_{k+1} - \alpha m_k}{1 + \lambda(k+1)} + (-\lambda\alpha) \frac{m_k}{(1 + \lambda k)(1 + \lambda(k+1))} \quad (3.29)$$

and

$$(m_{j+k})_{j,k=0}^n, (m_{j+k+1} - \alpha m_{j+k})_{j,k=0}^n, \quad n = 0, 1, 2, \dots \quad (3.30)$$

are positive definite, by the property of Schur product one obtains the matrices of (3.27) and (3.28) are positive definite.

Case 2. $-\infty < \alpha \leq 0$, $0 < \beta < +\infty$ (or $-\infty < \alpha < 0$, $0 \leq \beta < +\infty$ similarly). $J = I$.

Any non-negative polynomial $P(u)$ on $I = (\alpha, \beta)$ can be represented by

$$P(u) = P_1(u)^2 + (u - \alpha)(\beta - u)P_2(u)^2 \quad (3.31)$$

where $P_1(u)$ and $P_2(u)$ are polynomials with real coefficients.

In this case the conditions of (3.8) are equivalent to the matrices

$$(M_{j+k})_{j,k=0}^n, \quad n = 0, 1, 2, \dots \quad (3.32)$$

and

$$(-M_{j+k+2} + (\beta - \alpha)M_{j+k+1} - \alpha\beta M_{j+k})_{j,k=0}^n, \quad n = 0, 1, 2, \dots \quad (3.33)$$

are positive definite.

Note that $-M_{k+2} + (\beta - \alpha)M_{k+1} - \alpha\beta M_k$ can be rewritten by

$$\begin{aligned} \frac{-m_{k+2} + (\beta - \alpha)m_{k+1} - \alpha\beta m_k}{1 + \lambda(k+1)} + (-\alpha\beta) \frac{m_k}{(1 + \lambda k)(1 + \lambda(k+1))} \\ + \lambda \frac{m_{k+2}}{(1 + \lambda(k+1))(1 + \lambda(k+2))} \end{aligned} \quad (3.34)$$

and

$$(m_{j+k})_{j,k=0}^n, (-m_{k+2} + (\beta - \alpha)m_{k+1} - \alpha\beta m_k)_{j,k=0}^n, \quad n = 0, 1, 2, \dots \quad (3.35)$$

are positive definite, again by the property of Schur product one obtains the matrices of (3.32) and (3.33) are positive definite.

Remark 3.1 If $\alpha\beta > 0$, we cannot make sure that the matrices of (3.33) are positive, e.g., $\alpha = 1, \beta = 2, n = 0$

$$-M_2 + (\beta - \alpha)M_1 - \alpha\beta M_0 = -\frac{m_2}{1 + 2\lambda} + \frac{m_1}{1 + \lambda} - 2m_0 < 0 \quad (3.36)$$

for the sufficiently large λ .

Case 3. $0 < \alpha < \beta$ (or $\alpha < \beta < 0$ similarly). $J = (0, \beta)$ (or $J = (\alpha, 0)$).

We take $f(x)$ for the density function on $(0, \beta)$

$$\tilde{f}(x) = \begin{cases} f(x) & \alpha < x < \beta \\ 0 & 0 < x \leq \alpha \end{cases} \quad (3.37)$$

and by using Case 1 and Case 2 it is obvious.

Anyway, we get

Theorem 3.3 A probability density $\sigma(x)$ with its k th moment M_k of (3.2) exists, and is uniquely determined by the following differential equation

(1) $\alpha \leq 0 \leq \beta$,

$$\sigma(x) - \lambda[x\sigma(x)]^{(1)} = f(x), \quad x \in I; \quad (3.38)$$

(2) $\alpha < \beta < 0$,

$$\sigma(x) - \lambda[x\sigma(x)]^{(1)} = f(x)\chi_I, \quad x \in (\alpha, 0); \quad (3.39)$$

(3) $0 < \alpha < \beta$,

$$\sigma(x) - \lambda[x\sigma(x)]^{(1)} = f(x)\chi_I, \quad x \in (0, \beta). \quad (3.40)$$

4. PROOFS OF THEOREMS 1.2 AND 1.5

Proof of Theorem 1.2.

In [Wig1] Wigner made use of the integral representation of Bessel function of order 1 (pointed out by W.Feller to him) to get his semicircle law. However, in the original analysis he got the semicircle law by leading a differential equation. In the following we will combine these two kinds of method to derive the density.

Uniqueness: Note that using (1.33) $\sum_{j=0}^k C_{2k}^j C_{2k-j}^j \leq 3^{2k}$ holds. Writing $B = a + |b|$, then we have

$$M_{2k} \leq \sum_{j=0}^k C_{2k}^j C_{2k-j}^j a^{2j} b^{2k-2j} \leq (3B)^{2k}. \quad (4.1)$$

Thus, the Carleman's condition is satisfied and the density function is determined by the moments.

Derivation of the differential equation: Putting $2a = 1$, using the integral representation of Bessel function of order zero (see [Sze]) and calculating directly

$$\begin{aligned} \int_{-\infty}^{+\infty} (1 + \lambda itx) e^{itx} \sigma(x) dx &= \sum_{k=0}^{\infty} \frac{(it)^k}{k!} (1 + \lambda k) M_k \\ &= \sum_{k=0}^{\infty} \frac{(it)^k}{k!} \sum_{j=0}^{[k/2]} C_k^j C_{k-j}^j a^{2j} b^{k-2j} \\ &= \sum_{k=0}^{\infty} \frac{(it)^k}{k!} L_0 \left(az + \frac{a}{z} + b \right)^k \\ &= L_0 \sum_{k=0}^{\infty} \frac{(it)^k}{k!} \left(az + \frac{a}{z} + b \right)^k \\ &= L_0 \exp \left(it \left(az + \frac{a}{z} + b \right) \right) \\ &= e^{itb} L_0 \exp \left(it \left(az + \frac{a}{z} \right) \right) \\ &= e^{itb} \sum_{k=0}^{\infty} \frac{(ita)^{2k}}{(2k)!} C_{2k}^k \\ &= e^{itb} J_0(2at) \\ &= e^{itb} \frac{1}{\pi} \int_{-1}^1 \frac{e^{itx}}{\sqrt{1-x^2}} dx \\ &= \int_{b-1}^{b+1} e^{itx} f_b(x) dx \end{aligned} \quad (4.2)$$

where

$$f_b(x) = \frac{1}{\pi} \frac{1}{\sqrt{1 - (x - b)^2}}. \quad (4.3)$$

By using Theorems 3.2 and 3.3 in Section 3 one obtains

$$\sigma(x) - \lambda [x\sigma(x)]^{(1)} = f_b(x) \chi_{I_b} \quad (4.4)$$

and $\text{supp}(\sigma) = [B_1, B_2]$ for $\lambda > 0$ while for $\lambda = 0$, obviously

$$\sigma(x) = \frac{1}{\pi} \frac{1}{\sqrt{1 - (x - b)^2}}. \quad (4.5)$$

Now we give an exact solution of the equation of (4.4). Note that when $\lambda > 0$ the equation of (4.4) is known as a Cauchy-Euler equation of order 1 (see [GN], P99). Thus we have

(1) $-1 < b < 1$. 0 is a singular point of the equation of (4.4), and we have to determine how the solutions for $x < 0$ and $x > 0$ can be pieced together to give solutions valid on the whole interval I_b (see [GN], P22). So one obtains

$$\sigma(x) = \begin{cases} \frac{1}{\lambda} x^{\frac{1}{\lambda}-1} \int_x^{b+1} s^{-\frac{1}{\lambda}} f_b(s) ds & x > 0, \\ \frac{1}{\lambda} (-x)^{\frac{1}{\lambda}-1} \int_{b-1}^x (-s)^{-\frac{1}{\lambda}} f_b(s) ds & x < 0. \end{cases} \quad (4.6)$$

Note that $x\sigma(x)$ is absolutely continuous on I_b . Thus $\int_{b-1}^{b+1} \sigma(x) dx = 1$ from the equation of (4.4) by using $\sigma(b-1) = \sigma(b+1) = 0$. Besides, $\sigma(x)$ is continuous on I_b when $0 < \lambda < 1$, while 0 is a singular point of $\sigma(x)$ when $\lambda \geq 1$.

(2) $b = \pm 1$.

If $b = 1$, then

$$\sigma(x) = \frac{1}{\lambda} x^{\frac{1}{\lambda}-1} \int_x^2 s^{-\frac{1}{\lambda}} \frac{1}{\sqrt{s(2-s)}} ds, \quad x \in (0, 2). \quad (4.7)$$

While $b = -1$, we have

$$\sigma(x) = \frac{1}{\lambda} (-x)^{\frac{1}{\lambda}-1} \int_{-2}^x (-s)^{-\frac{1}{\lambda}} \frac{1}{\sqrt{-s(s+2)}} ds, \quad x \in (-2, 0). \quad (4.8)$$

(3) $b > 1$. The equation of (4.4) can be rewritten by one nonhomogeneous linear equation on $I_b = (b-1, b+1)$

$$\sigma(x) - \lambda [x\sigma(x)]^{(1)} = \frac{1}{\sqrt{1-(x-b)^2}} \quad (4.9)$$

and the other homogeneous linear equation on $(0, b-1)$

$$\sigma(x) - \lambda [x\sigma(x)]^{(1)} = 0. \quad (4.10)$$

Next we solve the related nonhomogeneous linear equation of (4.9)

$$\sigma(x) = \frac{1}{\lambda} x^{\frac{1}{\lambda}-1} \int_x^{b+1} s^{-\frac{1}{\lambda}} f_b(s) ds \quad (4.11)$$

and the related homogeneous linear equation of (4.10)

$$\sigma(x) = C_+ x^{\frac{1}{\lambda}-1} \quad (4.12)$$

where

$$C_+ = \frac{1}{\lambda} \int_{b-1}^{b+1} s^{-\frac{1}{\lambda}} f_b(s) ds. \quad (4.13)$$

Note that C_+ assures that $\sigma(x)$ is continuous on $(0, b+1)$ and $\int_0^{b+1} \sigma(x) dx = 1$.

(4) $b < -1$. Similar to the case where $b > 1$. One obtains the solutions of the equation of (4.4) on $I_b = (b-1, b+1)$

$$\sigma(x) = \frac{1}{\lambda} (-x)^{\frac{1}{\lambda}-1} \int_{b-1}^x (-s)^{-\frac{1}{\lambda}} f_b(s) ds \quad (4.14)$$

and on $(b+1, 0)$

$$\sigma(x) = C_- (-x)^{\frac{1}{\lambda}-1} \quad (4.15)$$

where

$$C_- = \frac{1}{\lambda} \int_{b-1}^{b+1} (-s)^{-\frac{1}{\lambda}} f_b(s) ds. \quad (4.16)$$

In the end we will complete the proof of Theorem 1.5. First, setting

$$\mathcal{H}_n = \text{span}\{p_0(x), p_1(x), \dots, p_{n-1}(x)\}, \quad (4.17)$$

then \mathcal{H}_n is an n -dimensional subspace of $L^2(\omega)$. It is obvious that $\hat{p}_0(x)p(x), \dots, \hat{p}_{n-l-1}(x)p(x)$ is a family of normalized orthogonal vectors in \mathcal{H}_n , extended by

$$e_0^{(n)}(x), \dots, e_{l-1}^{(n)}(x), \hat{p}_0(x)p(x), \dots, \hat{p}_{n-l-1}(x)p(x) \quad (4.18)$$

to a normalized orthogonal base of \mathcal{H}_n .

Let P_n be a projective operator from $L^2(\omega)$ to \mathcal{H}_n . We construct an operator from \mathcal{H}_n to itself as follows,

$$T_n^{(k)} = P_n \circ A_x^k : \mathcal{H}_n \longrightarrow \mathcal{H}_n \quad (4.19)$$

where A_x is the multiplication by x .

Lemma 4.1 Denote the k th moments of $\hat{\sigma}_N(x)$ and $\sigma_N(x)$ by $M_k^{(N)}$ and $\hat{M}_k^{(N)}$ respectively, then

$$\hat{M}_k^{(N)} = M_k^{(N)} + \Theta_N \quad (4.20)$$

where

$$\Theta_N = \frac{1}{N (c_N)^k} \left(\sum_{j=N-l}^{N-1} \langle A_x^k(\hat{p}_j p), \hat{p}_j p \rangle_{L^2(\omega)} - \sum_{j=0}^{l-1} \langle T_N^{(k)}(e_j^{(N)}), e_j^{(N)} \rangle_{L^2(\omega)} \right). \quad (4.21)$$

Proof. We first point out that

$$\begin{aligned} M_k^{(N)} &= \int x^k \sigma_N(x) dx \\ &= \frac{1}{N (c_N)^k} \sum_{j=0}^{N-1} \int_I x^k p_j^2(x) \omega(x) dx \\ &= \frac{1}{N (c_N)^k} \sum_{j=0}^{N-1} \langle x^k p_j, p_j \rangle_{L^2(\omega)} \\ &= \frac{1}{N (c_N)^k} \sum_{j=0}^{N-1} \langle T_N^{(k)}(p_j), p_j \rangle_{L^2(\omega)} \\ &= \frac{\text{Tr}(T_N^{(k)})}{N (c_N)^k}. \end{aligned} \quad (4.22)$$

On the other hand, by the normalized orthogonal base of (4.18) one obtains

$$\text{Tr}(T_N^{(k)}) = \sum_{j=0}^{N-l-1} \langle T_N^{(k)}(\hat{p}_j p), \hat{p}_j p \rangle_{L^2(\omega)} + \sum_{j=0}^{l-1} \langle T_N^{(k)}(e_j^{(N)}), e_j^{(N)} \rangle_{L^2(\omega)}. \quad (4.23)$$

Thus,

$$\begin{aligned}
\hat{M}_k^{(N)} &= \int x^k \hat{\sigma}_N(x) dx \\
&= \frac{1}{N (c_N)^k} \sum_{j=0}^{N-1} \int_I x^k \hat{p}_j^2(x) p^2(x) \omega(x) dx \\
&= \frac{1}{N (c_N)^k} \left(\sum_{j=0}^{N-l-1} \left\langle \int_I x^k \hat{p}_j^2(x) p^2(x) \omega(x) dx + \sum_{j=0}^{l-1} \int_I x^k (e_j^{(N)}(x))^2 \omega(x) dx \right\rangle + \Theta_N \right) \\
&= \frac{1}{N (c_N)^k} \left(\sum_{j=0}^{N-l-1} \langle T_N^{(k)}(\hat{p}_j p), \hat{p}_j p \rangle_{L^2(\omega)} + \sum_{j=0}^{l-1} \langle T_N^{(k)}(e_j^{(N)}), e_j^{(N)} \rangle_{L^2(\omega)} \right) + \Theta_N \\
&= M_k^{(N)} + \Theta_N.
\end{aligned} \tag{4.24}$$

□

Lemma 4.2 Write $\|\cdot\| = \langle \cdot, \cdot \rangle_{L^2(\omega)}^{1/2}$, and for $k = 1, 2, \dots$,

$$\|A_x^k f\| \leq 3^k \left(\prod_{j=0}^{k-1} D_{n+j} \right) \|f\|, \quad \forall f \in \mathcal{H}_n \tag{4.25}$$

where

$$D_n = \max_{0 \leq j \leq n} \{a_j, |b_j|\}. \tag{4.26}$$

Proof. Set $f(x) = \sum_{j=0}^{N-1} l_j p_j(x)$ and note that

$$\begin{aligned}
\|A_x f\|^2 &= \left\| \sum_{j=0}^{N-1} l_j x p_j \right\|^2 \\
&= \left\| \sum_{j=0}^{N-1} l_j (a_{j+1} p_{j+1} + b_j p_j + a_j p_{j-1}) \right\|^2 \\
&= \left\| \sum_j (l_{j-1} a_j + l_j b_j + l_{j+1} a_{j+1}) p_j \right\|^2 \\
&= \sum_j |l_{j-1} a_j + l_j b_j + l_{j+1} a_{j+1}|^2 \\
&\leq 3D_n^2 \sum_j (l_{j-1}^2 + l_j^2 + l_{j+1}^2) \\
&\leq 9D_n^2 \|f\|^2.
\end{aligned} \tag{4.27}$$

Thus, $\|A_x f\| \leq 3D_n \|f\|$ and (4.25) is easily proved. □

Proof of Theorem 1.5: By (1.21) and Lemma 4.2, there exist constants C_1 and C_2 which only depend on k such that

$$|\langle A_x^k(\hat{p}_j p), \hat{p}_j p \rangle_{L^2(\omega)}| \leq \|A_x^k(\hat{p}_j p)\| \leq 3^k D_{N+l+k-2}^k \leq C_1 (c_N)^k \tag{4.28}$$

and

$$|\langle T_N^{(k)}(e_j^{(N)}), e_j^{(N)} \rangle_{L^2(\omega)}| \leq \|P_N\| \|A_x^k e_j^{(N)}\| \leq 3^k D_{N+k-1}^k \leq C_2(c_N)^k. \quad (4.29)$$

Thus, using (4.21), for the large N ,

$$|\Theta_N| \leq \frac{1}{N (c_N)^k} \left(\sum_{j=N-l}^{N-1} C_1(c_N)^k + \sum_{j=0}^{l-1} C_2(c_N)^k \right) = \frac{(C_1 + C_2)l}{N}. \quad (4.30)$$

Again by Lemma 4.1 we get

$$\lim_{n \rightarrow \infty} \hat{M}_k^{(N)} = \lim_{n \rightarrow \infty} M_k^{(N)} = M_k. \quad (4.31)$$

□

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